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Estimation of Point-to-Point Telephone Traffic

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Estimates of point-to-point telephone traffic are required for the current and the long-range planning of the Bell System's Public Switched Network. Because of the potentially immense volume of data which must be processed, these estimates are typically based upon small samples of total traffic and, therefore, can have large statistical errors. In this paper, we develop a model for quantifying the accuracy of point-to-point traffic measurements as a function of sample size and traffic parameters. Together with a worth-of-data model, not described here, our results can be used to establish a cost-optimal sampling rate for point-to-point traffic measurement systems. However, our results have been used to establish 20 percent as an upper bound on a cost-optimal sampling rate for a usage measurement system and 10 percent for an attempt-only measurement system. We show, however, that the attempt-based estimate is, for sampling rates greater than about 2 percent, less accurate than the usage-based estimate. We also show how the accuracy of point-to-point load estimates can be improved by employing a ratio-estimate which combines point-to-point and trunk-group measurements; however, in practical applications, we find that the improvement is not significant.

I. INTRODUCTION

Trunk-group and point-to-point traffic data systems provide the measurements of telephone traffic which are used for the current and the long-range planning of the Bell System's Public Switched Network. Trunk-group data systems provide estimates of the traffic offered to existing trunk groups. Normally, an estimate of trunk-group offered load is based upon a direct measurement of the average number of busy trunks, the average attempt count, and the average overflow count.[1]

Point-to-point traffic data systems provide estimates of the telephone traffic which originates at one and terminates at the other of a specific pair of network points not necessarily joined by a single trunk group; for

example, the end-office pair (A_1,B_1) of Fig. 1. In the trunk-provisioning process, estimates of point-to-point offered loads are required to plan for the introduction of new trunk groups and the rehoming of end-offices or tandems. In general, they are also used, as a supplement to trunk-group measurements, in the network disassembly process (the process that converts measured loads on trunk groups which receive overflow traffic to first-route loads) and in the network assembly process (the process which converts projected first-route loads to total offered loads). Moreover, with the possible introduction of dynamic traffic routing, our studies have shown that the trunk-provisioning process will require more extensive use of point-to-point data tban is required in the present hierarchical fixed-routing network.

Estimates of point-to-point offered loads cannot, in general, be derived from trunk group measurements since trunk groups typically carry more than one point-to-point load. Instead, estimates of point-to-point offered loads are derived from detailed records of the origin, destination, and, when available, holding times of individual calls. (When holding times are not available, a load estimate can be based upon an attempt count measurement together with an exogenous estimate of mean holding time; see Section 3.2.)

To reduce the costs for recording and processing point-to-point data, most existing measurement systems have been designed to record only a small sample of total traffic. For example, the Centralized Message Data System (CMDS, see Section II) provides estimates of point-to-point loads derived from a 5-percent sample of all toll calls. But while sampling reduces the cost of providing point-to-point data, it also introduces statistical measurement errors that reduce the accuracy and, hence, the worth of the data.

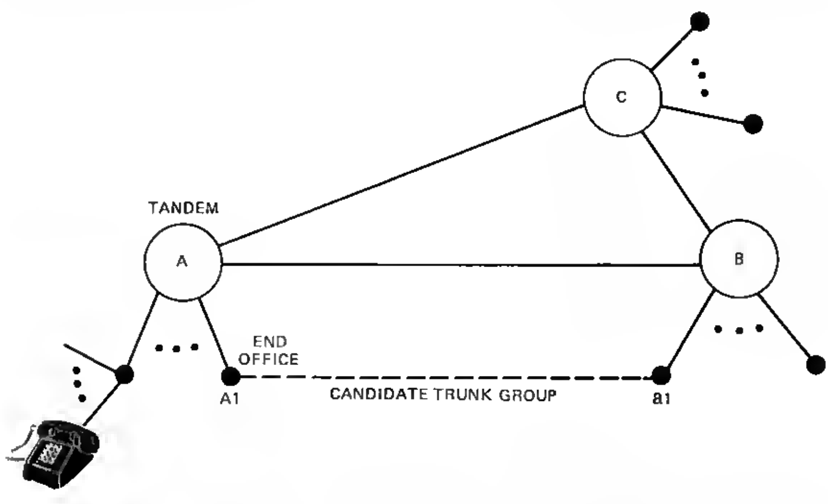

Fig. 1—An application of point-to-point data: planning new trunk groups.

In this paper, we develop a model for quantifying the accuracy of point-to-point traffic measurements as a function of sample size and traffic parameters. Together with a worth-of-data model[2] for quantifying the cost impact of data errors on the network provisioning process, this data accuracy model can be used to establish the trunk-engineering requirements for point-to-point data systems.

In Section II, we describe how point-to-point loads are measured by CMDS and we develop a model for quantifying sampling error. While this is a specific example, the methods and results are directly applicable to other (existing and proposed) point-to-point traffic measurement systems. In Section III we use our model to analyze three methods for estimating point-to-point loads: one based upon a usage measurement, one upon an attempt count together with an exogenous estimate of holding time, and one upon a combination of point-to-point and trunk-group measurements. A summary is given in the last section, and the required statistical results are developed in Appendices A and B.

II. POINT-TO-POINT MEASUREMENTS

For toll traffic, the major source of point-to-point data is provided by the Centralized Message Data System. In this section, we describe the CMDS data base and model the various sources of error.

2.1 The CMDS data base

For every point-to-point traffic item (defined by originating and terminating end-office prefix codes), the CMDS data base provides an *estimate* of both the total number of calls and the associated usage (i.e., sum of holding times) for calls that originate during a time-consistent hour over 20 consecutive business days.

These estimates are based upon a 5-percent sample of the total number of calls processed by the toll billing equipment in each Regional Accounting Office (RAO). Figure 2 illustrates the process. Automatic Message Accounting (AMA) tapes are periodically shipped to a Regional Accounting Office where they are processed to produce sequential records of the origin, destination, and conversation time of individual calls. As these records are processed for customer billing, the record for every 20th call is transmitted (in a batch mode) to the CMDS computer in Kansas City, where they are sorted and summarized to provide estimates of individual point-to-point loads.

2.2 Sources of error

Since estimates of point-to-point offered loads are based upon measurements made over several time-consistent hours, and since source loads are known to vary from day to day, our model will account for statistical errors due to both the finite measurement interval and day-

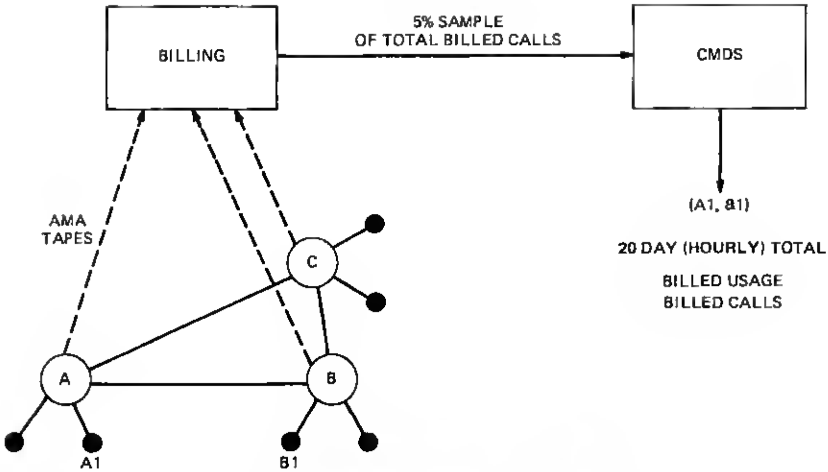

Fig. 2—Centralized message data system.

to-day load variation.[3] Furthermore, since the measurements are obtained from a 5-percent sample of total traffic, our model will also account for variations in the sample size for individual point-pairs. That is, depending upon the position of calls in the sequence of message records (from which the 5-percent sample is obtained) the actual sample size for an individual point-pair can be more, or less, than 5 percent. In Section 2.3, we develop a model for quantifying these sources of error.

The CMDS data base excludes toll traffic which is not billed. In addition, of course, to blocked calls, CMDS also excludes call set-up and ringing time (for both completed and noncompleted calls), directory assistance calls, and official calls which are not detailed billed. Estimates of this nonbilled usage are, therefore, an additional source of error for CMDS-based load estimates. However, our studies have shown that this error is negligible in comparison with sampling error and, therefore, it will not be accounted for by our model. (Section 3.2 describes a method for estimating nonbilled usage.)

2.3 Mathematical model

Estimates of point-to-point offered loads are normally based upon measurements made over K disjoint time-consistent intervals $I_1, \ldots, I_K$, each of length t (typically, $K = 20$ and $t = 1$ hour). We assume that the distribution of realized loads can be described by the model used by Hill and Neal[3] to explain the observed variation of trunk-group offered loads. Thus, during I_j, we assume that call arrivals are Poisson-distributed* with rate λ_j and that call-holding times are independent and exponen-

* Point-to-point offered loads correspond to trunk-group first-offered (Poisson) loads; hence, it is appropriate to set the peakedness factor, z, of Ref. 3 to unity.

tially distributed with mean h. Furthermore, in accordance with the model for day-to-day load variation developed in Ref. 3, the loads $\alpha_i = \lambda_i h$, $i = 1,\ldots,K$, are assumed to be independent and identically distributed with mean $a = \lambda h$ and variance

$$v_d = \max\left\{0, 0.13a^{\phi} - \frac{2a}{t/h}\right\}, \tag{1}$$

where ϕ is a parameter that describes the level of day-to-day variation. For engineering applications, we use $\phi = 1.5$, 1.7, or 1.84, which are referred to, respectively, as low, medium, or high day-to-day variation. For first-routed and point-to-point traffic, $\phi = 1.5$ is usually appropriate.

To model the sampling process, we assume that in the sequence of message records, each call associated with a given point-pair is included in the sample with the same probability p (for CMDS, $p = 0.05$); i.e., we assume a multinomial distribution for the numbers of sampled calls belonging to given point-pairs. (For CMDS, the actual distribution is more closely approximated by a hypergeometric distribution; however, since the number of calls belonging to a given point-pair is a small fraction of the total number of calls processed by an RAO, our simplifying assumption introduces no significant loss of accuracy.)

Let N_j denote the number of arrivals during I_j and let h_{ij} be the holding time of the ith arrival in I_j. Then, with $\delta_{ij} = 1$ if the ith call is included in the sample and zero otherwise,

$$c = \sum_{j=1}^{K} \sum_{i=1}^{Nj} \delta_{ij} \tag{2}$$

is the total number of sampled calls during $I = \Sigma_{j=1}^{K} I_j$, and

$$u = \sum_{j=1}^{K} \sum_{i=1}^{Nj} h_{ij}\delta_{ij} \tag{3}$$

is the corresponding usage.

III. LOAD ESTIMATES

In this section, we analyze three procedures for estimating point-to-point loads. The first estimate, $\hat{a}^{(1)}$, is based upon the usage measurement, u; the second estimate, $\hat{a}^{(2)}$, upon the attempt count, c; and the third estimate, $\hat{a}^{(3)}$, upon a combination of point-to-point and trunk-group measurements. (Although these do not exhaust the possible estimates, they do form the basis for analyzing more complex estimates; for example, an estimate of the offered load at 10 a.m. could be based upon a combination of the measured loads at 9, 10, and 11 a.m.) In each case, we use mean square error (MSE) to measure the accuracy of the load estimate, i.e., if $\hat{a}$ denotes an estimate of the mean offered load, then

$$\begin{aligned} \mathrm{MSE}\{\hat{a}\} &= E\{\hat{a} - a\}^2 \\ &= \mathrm{Var}\{\hat{a}\} + E^2\{\hat{a} - a\}. \end{aligned} \tag{4}$$

3.1 Estimate 1

Since u [eq. (3)] is a p-sample of usage over K intervals each of length t,

$$\hat{a}^{(1)} = \frac{1}{Kpt} u \tag{5}$$

is an estimate of the corresponding average offered load a.

In Appendix A, we show that $\hat{a}^{(1)}$ is unbiased, i.e.,

$$E\{\hat{a}^{(1)}\} = a, \tag{6}$$

and has variance

$$\mathrm{Var}\{\hat{a}^{(1)}\} = \frac{1}{K}\left\{\frac{2a}{pt/h} + v_d\right\}; \tag{7}$$

hence, from eq. (4),

$$\mathrm{MSE}\{\hat{a}^{(1)}\} = \frac{1}{K}\left\{\frac{2a}{pt/h} + v_d\right\}. \tag{8}$$

In (8), the first term $\{2a/pt/h\}$ represents the combined effects of the finite measurement interval and deviations from the average sample size. The second term $\{v_d\}$ is due to (day-to-day) variations in the source load. Of course, the factor K is due to averaging measurements over K independent intervals.

Figure 3 displays the root-mean-square (RMS) error of $\hat{a}^{(1)}$ (in percent of mean load) as a function of average offered load for sampling rates of 5 and 100 percent. The results for a 5-percent sample apply when the offered load is estimated using CMDS data, while those for a 100-percent

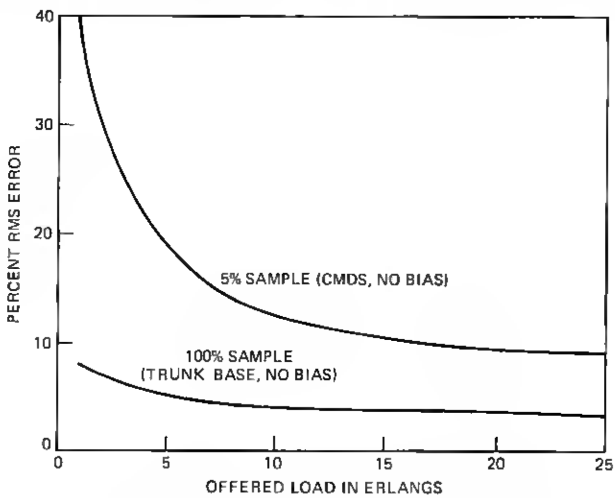

Fig. 3—Sampling error vs offered load.

sample apply when the load estimate is based directly upon trunk-group measurements. As noted in Section 2.2, errors in estimates of nonbilled usage, associated with CMDS estimates, are negligible in comparison with sampling error. Also, in applying our results to trunk-group measurements, we assume that u [eq. (3)] adequately approximates the actual usage *during* the measurement interval I; i.e., we assume that the edge effects are negligible (see Ref. 3). Furthermore, our studies have shown that the additional variance caused by discretely sampling the usage with a 100-second-scan Traffic Usage Recorder is negligible when compared with the variance caused by day-to-day load variation. The results shown in Fig. 3 assume the standard measurement interval (t = 1 hour, K = 20), low day-to-day variation (ϕ = 1.5), and h = 250 seconds.

Note that estimates based upon a 5-percent sample can have errors that are large relative to those based upon a 100-percent sample. For example, for an offered load of 5 erlangs (typical of base year prove-in loads for new high-usage trunk groups), the RMS error for a 5-percent sample is about 20 percent, compared with an RMS error of about 5 percent for a 100-percent sample. Similarly, for an offered load of about 15 erlangs (typical of loads offered to existing Long Lines high-usage trunk groups), an estimate based upon a 5-percent sample has an RMS error of about 12 percent, while for a 100-percent sample, the RMS error is about 4 percent.

Figure 4 displays the percent RMS error of $\hat{a}^{(1)}$ as a function of the sampling rate p for offered loads of 5 and 15 erlangs. The important result to note is that the statistical variability of $\hat{a}^{(1)}$ does not decrease appreciably as the sampling rate is increased beyond about 20 percent. This occurs since the contribution of day-to-day load variation is independent of the sampling rate, and above a sampling rate of about 20

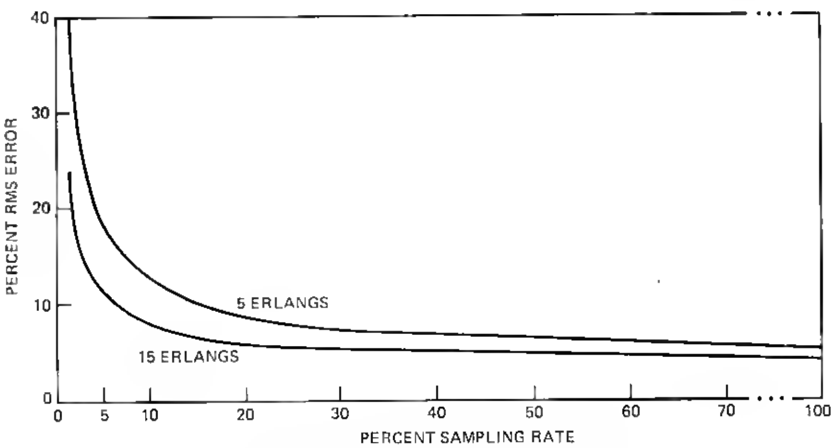

Fig. 4—Sampling error vs sampling rate.

percent it becomes the dominant source of error. Of course, any improvement in accuracy is significant if the associated benefits justify the increased cost for data collection and processing. However, using the worth-of-data model of Ref. 2, wbich quantifies the cost impact of data errors on the provisioning of direct final (i.e., nonalternate route) trunk groups, we have established 20 percent as an *upper bound* on a cost-optimal sampling rate. To estahlish an actual cost-optimal sampling rate, however, we require a worth-of-data model whicb applies to more general network configurations (i.e., alternate routing networks); such a model is currently being formulated.

3.2 Estimate 2

The usage-based estimate of offered load (i.e., Estimate 1) is derived from attempt count and holding-time measurements. In this section, we analyze an alternative estimate based upon an attempt count together with an exogenous estimate of the corresponding mean holding time. The data collection and processing costs for an attempt-based point-to-point data system are less than for a usage-hased system; however, we will show that the load estimates are suhstantially less accurate.

Let $\hat{h}$ (a constant) denote an estimate of the mean holding time h. Then, since c [eq. (2)] is the total number of sampled calls during I, c/Kpt is an estimate of the mean attempt rate λ and, therefore,

$$\hat{a}^{(2)} = \frac{1}{Kpt} c\hat{h} \tag{9}$$

is an estimate of the average offered load a.

In Appendix A we show that

$$E\{\hat{a}^{(2)}\} = \lambda\hat{h} \tag{10}$$

so that $\hat{a}^{(2)}$ is biased whenever $\hat{h} \neq h$, and

$$\mathrm{Var}\{\hat{a}^{(2)}\} = \left(\frac{\hat{h}}{h}\right)^2 \frac{1}{K} \left\{\frac{a}{pt/h} + v_d\right\}; \tag{11}$$

hence, from (4),

$$\mathrm{MSE}\{\hat{a}^{(2)}\} = \left(\frac{\hat{h}}{h}\right)^2 \frac{1}{K} \left\{\frac{a}{pt/h} + v_d\right\} + \lambda^2(\hat{h} - h)^2. \tag{12}$$

Clearly, $\mathrm{MSE}\{\hat{a}^{(2)}\}$ depends upon the error $(\hat{h} - h)$. In practice, the same estimate $\hat{h}$ would be applied to a collection of point-pairs (e.g., all point-pairs witbin an operating company, or all point-pairs served by a common trunk group), and our studies have found that the corresponding distribution of errors $(\hat{h} - h)$ has a coefficient of variation of at least 20 percent. Accordingly, our numerical results will assume that $\hat{h}$ is in error by 20 percent.

Figure 5 displays the percent RMS error of $\hat{a}^{(2)}$ as a function of sampling rate for an offered load of 5 erlangs. We assume the same numerical values for K, t, ϕ, and h as in Fig. 3, and we assume $\hat{h}/h = 1.2$. For purposes of comparison, Fig. 5 also displays the percent RMS error of $\hat{a}^{(1)}$, as given previously in Fig. 4.

We draw two conclusions from the results shown in Fig. 5. First, whereas 20 percent is a reasonable upper bound on sampling rate for a usage-based measurement system, a sampling rate of about 10 percent is sufficient for an attempt-based system. Of course, if $\hat{h}$ were known to be in error by more (less) than 20 percent, a sampling rate of less (more) than 10 percent would be appropriate. But if we know only the coefficient of variation of the distribution of h (which we assume to be 20 percent), then the average value of $\text{MSE}\{\hat{a}^{(2)}\}$, with respect to this distribution, cannot be significantly reduced by increasing the sampling rate beyond 10 percent. Second, we note that an estimate based upon measured usage is, for sampling rates greater than about 2 percent, more accurate than an estimate based upon an attempt count. (For sampling rates less than 2 percent, the standard deviation of the measured holding time exceeds that of the estimate $\hat{h}$; hence, $\hat{a}^{(2)}$ is relatively more accurate in this range.)

In view of the above results, we conclude that usage measurements (when available) are preferable to attempt counts for estimating point-to-point loads. However, our studies have shown that the attempt count provides a more accurate basis for estimating CMDS nonbilled usage than does the measured (billed) usage. That is, with CMDS data, an estimate of the form $\hat{a} = (u + \hat{\beta}c)/Kpt$ is employed, where the first term $\{u/Kpt\}$ is an estimate of billed load and the second term $\{\hat{\beta}c/Kpt\}$ is an estimate of nonbilled load. Thus, $\hat{\beta}$ can be interpreted as an estimate

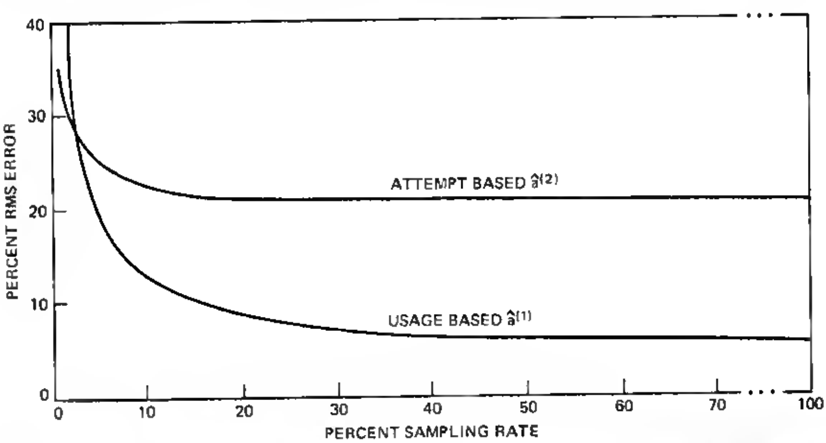

Fig. 5—Comparison of attempt-based and usage-based offered-load estimates (offered load = 5 erlangs).

of an average nonbilled holding time per billed attempt. Furthermore, we have shown that a small additional improvement can be obtained by employing a load-dependent combination of u and c to estimate billed load. However, this additional improvement is not significant.

3.3 Estimate 3

In this section, we show how the statistical variability of point-to-point load estimates can be reduced by combining point-to-point and trunk-group measurements. The procedure we describe has been proposed as a means for improving the accuracy of CMDS-based load estimates; however, we show that the improvement is not significant.

Consider a trunk group whose total offered load is the sum of N point-to-point first-offered loads. For the ith load, $i = 1,\ldots,N$, let a_i denote the mean load and let $\hat{a}_i^{(1)}$ be the (point-to-point) usage-based estimate of a_i. Furthermore, let $\hat{T}$ denote the estimate of trunk-group offered load based upon trunk-group usage measurements and let $\hat{A} = \Sigma_{i=1}^{N} \hat{a}_i^{(1)}$ denote the corresponding estimate (for the same measurement interval) based upon point-to-point usage data. Since $\hat{T}$ is based upon a 100-percent sample, the difference $(\hat{T} - \hat{A})$ measures the sum of the errors relative to the realized loads in the individual estimates $\hat{a}_i^{(1)}$. By assigning a fraction (w_i) of this difference to the individual estimates $\hat{a}_i^{(1)}$, we obtain a new estimate of a_i; i.e.,

$$\hat{a}_i^{(3)} = \hat{a}_i^{(1)} + w_i(\hat{T} - \hat{A}). \tag{13}$$

In Appendix B, we show that an approximation to a minimum-variance linear estimate of a_i is obtained when $w_i = \hat{a}_i^{(1)}/\hat{A}$. Thus, we have the ratio-estimate

$$\hat{a}_i^{(3)} = \frac{\hat{a}_i^{(1)}}{\hat{A}} \hat{T}. \tag{14}$$

Since $\hat{a}_i^{(1)}$ appears as a summand in $\hat{A}$, the ratio $\hat{T}/\hat{A}$ is negatively correlated (or tends to vary inversely) with $\hat{a}_i^{(1)}$. Physically, it is this negative correlation which makes $\hat{a}_i^{(3)}$ statistically less variable than $\hat{a}_i^{(1)}$.

By employing a first-order Taylor series approximation to $\hat{a}_i^{(3)}$, we obtain in Appendix A the following approximations for the mean and variance of $\hat{a}_i^{(3)}$:

$$E\{\hat{a}_i^{(3)}\} \approx a_i \tag{15}$$

and

$$\mathrm{Var}\{\hat{a}_i^{(3)}\} \approx \frac{2a_i}{Kpt/h}\{1 - f_i(1-p)\} + \frac{1}{K} v_{di}, \tag{16}$$

where

$$f_i = \frac{a_i}{\sum_{j=1}^{N} a_j} \tag{17}$$

is the fraction of the total load contributed by the ith point-pair. Also, since our results are not significantly affected by differences in mean holding times, we have assumed that each point-to-point load has the same mean holding-time, h. From (4), (15), and (16) we have

$$\mathrm{MSE}\{\hat{a}_i^{(3)}\} \approx \frac{2a_i}{Kpt/h}\{1 - f_i(1-p)\} + \frac{1}{K}v_{di}. \tag{18}$$

Figure 6 displays the percent RMS error of $\hat{a}_i^{(3)}$ as a function of offered load for several values of the parameter f_i. We assume a sampling rate of 5 percent (the CMDS sampling rate) and the same numerical values for K, t, h, and ϕ as in Fig. 3. Note that $\hat{a}_i^{(3)}$ is more accurate than $\hat{a}_i^{(1)}$ and that the relative difference in accuracy is a maximum when f_i equals one (since $\hat{a}_i^{(3)} = \hat{T}$ when $f_i = 1$) and approaches zero as f_i approaches zero (since the variance of $\hat{T}/\hat{A}$ approaches zero and, hence, $\hat{a}_i^{(3)}$ approaches $\hat{a}_i^{(1)}$ as f_i approaches zero).

The results of Fig. 6 are perhaps more striking when viewed in terms of the reciprocal of f_i, which can be interpreted as the number (N') of equal-sized point-to-point loads corresponding to f_i. That is, the relative difference in accuracy of $\hat{a}_i^{(1)}$ and $\hat{a}_i^{(3)}$ is a rapidly decreasing function of N'; for N' greater than 4, the relative difference is less than about 4 percentage points.

Typically, trunk groups carry a large number of point-to-point loads, each of which represents a small fraction ($f_i \ll 1$) of total offered load. In this region, Fig. 6 shows that the use of trunk-group measurements provides only a small improvement in the quality of CMDS-based load estimates. Again, any improvement is significant if the associated ben-

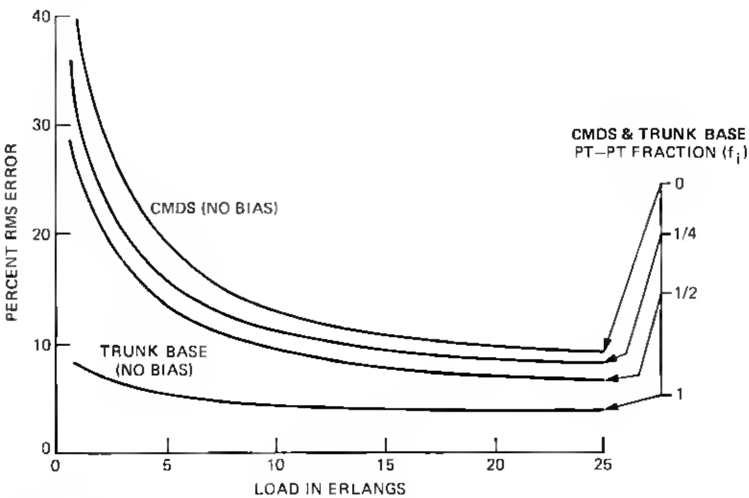

Fig. 6—Reduction in rms error afforded by combining point-to-point and trunk-base measurements.

efits justify the increased development and data processing costs. However, based upon the worth-of-data model of Ref. 2, we have concluded that employing trunk-group measurements will not significantly reduce the statistical errors associated with CMDS-based estimates of point-to-point loads.

IV. SUMMARY AND CONCLUSIONS

We have developed a model for quantifying the accuracy of point-to-point traffic measurements as a function of sampling rate and traffic parameters. Using this model, we have established 20 percent as an upper bound on a cost-optimal sampling rate for a usage-based measurement system and 10 percent for an attempt-based system. Furthermore, for sampling rates greater than a few percent and loads in the range of engineering interest, our results show that a usage-based load estimate is more accurate than an attempt-based load estimate. We also showed that the accuracy of (CMDS) load estimates could be improved by employing a ratio estimate that combines point-to-point and trunk-group measurements; however, in practical applications, the improvement is not significant. Our results, together with a worth-of-data model,[2] can be used to establish requirements for point-to-point traffic measurement systems.

APPENDIX A

Mean and Variance of Load Estimates

A.1 Estimate 1

From eqs. (3) and (5) and Ref. 4,

$$E\{\hat{a}^{(1)}\} = \frac{1}{Kpt} \sum_{j=1}^{K} E\left\{E\left\{\sum_{i=1}^{N_j} h_{ij}\delta_{ij} \,|\, N_j\right\}\right\}. \tag{19}$$

Since the h_{ij} and δ_{ij} are independent, and since arrivals during I_j are Poisson-distributed with rate λ_j, we have

$$\begin{aligned} E\left\{E\left\{\sum_{i=1}^{N_j} h_{ij}\delta_{ij} \,|\, N_j\right\}\right\} &= E\{N_j ph\} \\ &= phE\{E\{N_j \,|\, \lambda_j\}\} \\ &= phE\{\lambda_j t\} \\ &= ph\lambda t. \end{aligned} \tag{20}$$

Substituting (20) into (19) gives

$$\begin{aligned} E\{\hat{a}^{(1)}\} &= \lambda h \\ &= a. \end{aligned} \tag{21}$$

Furthermore, since the measurements during each interval I_j are uncorrelated, (3) and (5) give

$$\mathrm{Var}\{\hat{a}^{(1)}\} = \frac{1}{(Kpt)^2} \sum_{j=1}^{K} \mathrm{Var}\left\{\sum_{i=1}^{N_j} h_{ij}\delta_{ij}\right\}. \tag{22}$$

From Ref. 4,

$$\begin{aligned} \mathrm{Var}\left\{\sum_{i=1}^{N_j} h_{ij}\delta_{ij}\right\} &= E\left\{\mathrm{Var}\left\{\sum_{i=1}^{N_j} h_{ij}\delta_{ij} | N_j\right\}\right\} \\ &\quad + \mathrm{Var}\left\{E\left\{\sum_{i=1}^{N_j} h_{ij}\delta_{ij} | N_j\right\}\right\} \\ &= E\{N_j h^2 p(2-p)\} + \mathrm{Var}\{N_j ph\} \\ &= \lambda t h^2 p(2-p) + p^2 h^2 \mathrm{Var}\{N_j\}. \end{aligned} \tag{23}$$

Again, given λ_j, N_j is Poisson-distributed; hence, $\mathrm{Var}\{N_j | \lambda_j\} = E\{N_j | \lambda_j\} = \lambda_j t$.

Thus it follows that

$$\begin{aligned} \mathrm{Var}\{N_j\} &= E\{\mathrm{Var}\{N_j | \lambda_j\}\} \\ &\quad + \mathrm{Var}\{E\{N_j | \lambda_j\}\} \\ &= E\{\lambda_j t\} + \mathrm{Var}\{\lambda_j t\} \\ &= \lambda t + t^2 \mathrm{Var}\{\lambda_j\}. \end{aligned} \tag{24}$$

Substituting (24) into (23) gives

$$\mathrm{Var}\left\{\sum_{i=1}^{N_j} h_{ij}\delta_{ij}\right\} = 2\lambda t h^2 p + p^2 t^2 h^2 \mathrm{Var}\{\lambda_j\}. \tag{25}$$

The offered load during I_j is $\alpha_j = \lambda_j h$; hence, $v_d = \mathrm{Var}\{\alpha_j\} = h^2 \mathrm{Var}\{\lambda_j\}$.

Thus, from (22) and (25), we have

$$\mathrm{Var}\{\hat{a}^{(1)}\} = \frac{1}{K}\left\{\frac{2a}{pt/h} + v_d\right\}. \tag{26}$$

We now develop an expression, which we require in Section A.3, for

$$\mathrm{Cov}\{\hat{a}^{(1)}, \hat{a}^{(1)}|_{p=1}\},$$

where

$$\hat{a}^{(1)}|_{p=1} = \frac{1}{Kt} \sum_{j=1}^{K} \sum_{i=1}^{N_j} h_{ij} \tag{27}$$

corresponds to a sampling rate of 100 percent. Thus, from (5) and (27), we have

$$\mathrm{Cov}\{\hat{a}^{(1)}, \hat{a}^{(1)}|_{p=1}\} = \frac{1}{K^2 t^2 p} \sum_{j=1}^{K} \mathrm{Cov}\left\{\sum_{i=1}^{N_j} h_{ij}\delta_{ij}, \sum_{i=1}^{N_j} h_{ij}\right\}. \tag{28}$$

From Ref. 4,

$$\mathrm{Cov}\left\{\sum_{i=1}^{N_j} h_{ij}\delta_{ij}, \sum_{i=1}^{N_j} h_{ij}\right\} = E\left\{\mathrm{Cov}\left\{\sum_{i=1}^{N_j} h_{ij}\delta_{ij} | N_j, \sum_{i=1}^{N_j} h_{ij} | N_j\right\}\right\}$$

$$+ \mathrm{Cov}\left\{E\left\{\sum_{i=1}^{N_j} h_{ij}\delta_{ij}|N_j\right\}, E\left\{\sum_{i=1}^{N_j} h_{ij}|N_j\right\}\right\}. \quad (29)$$

We first expand

$$\mathrm{Cov}\left\{\sum_{i=1}^{N_j} h_{ij}\delta_{ij}|N_j, \sum_{i=1}^{N_j} h_{ij}|N_j\right\}$$

$$\triangleq E\left\{\left(\sum_{i=1}^{N_j} h_{ij}\delta_{ij}|N_j\right)\left(\sum_{i=1}^{N_j} h_{ij}|N_j\right)\right\} - E\left\{\sum_{i=1}^{N_j} h_{ij}\delta_{ij}|N_j\right\} E\left\{\sum_{i=1}^{N_j} h_{ij}|N_j\right\}$$

$$= N_j\{p2h^2 + (N_j - 1)ph^2\} - N_j^2 ph^2$$

$$= N_j ph^2. \quad (30)$$

Substituting (30) into (29) and using (24) gives

$$\mathrm{Cov}\left\{\sum_{i=1}^{N_j} h_{ij}\delta_{ij}, \sum_{i=1}^{N_j} h_{ij}\right\} = ph^2 E\{N_j\} + \mathrm{Cov}\{N_j ph, N_j h\}$$

$$= ph^2 E\{N_j\} + ph^2 \mathrm{Var}\{N_j\}$$

$$= 2phat + pt^2 v_d. \quad (31)$$

Thus, from (28) and (31), we have

$$\mathrm{Cov}\{\hat{a}^{(1)}, \hat{a}^{(1)}|_{p=1}\} = \frac{1}{K}\left\{\frac{2a}{t/h} + v_d\right\}. \quad (32)$$

A.2 Estimate 2

Using expansions similar to those of A.1 it follows, from (2) and (9), that

$$E\{\hat{a}^{(2)}\} = \frac{\hat{h}}{Kpt}\sum_{j=1}^{K} E\left\{E\left\{\sum_{i=1}^{N_j} \delta_{ij}|N_j\right\}\right\}$$

$$= \hat{h}\lambda \quad (33)$$

and

$$\mathrm{Var}\{\hat{a}^{(2)}\} = \left(\frac{\hat{h}}{Kpt}\right)^2 \sum_{j=1}^{K} \mathrm{Var}\left\{\sum_{i=1}^{N_j} \delta_{ij}\right\}$$

$$= \left(\frac{\hat{h}}{h}\right)^2 \frac{1}{K}\left\{\frac{a}{pt/h} + v_d\right\}. \quad (34)$$

A.3 Estimate 3

An approximation for the mean and variance of $\hat{a}_i^{(3)}$ is obtained by expanding the right-hand side of eq. (14) in a three-dimensional Taylor series about the point $\{E\{\hat{T}\}, E\{\hat{A}\}, E\{\hat{a}_i^{(1)}\}\}$. To first order, this gives the approximations

$$E\{\hat{a}_i^{(3)}\} \approx \frac{E\{\hat{T}\}}{E\{\hat{A}\}} E\{\hat{a}_i^{(1)}\} \quad (35)$$

and

$$\operatorname{Var}\{\hat{a}_i^{(3)}\} \approx \operatorname{Var}\{\hat{a}_i^{(1)}\} + f_i^2 \operatorname{Var}\{\hat{A}\} + f_i^2 \operatorname{Var}\{\hat{T}\} - 2f_i \operatorname{Cov}\{\hat{a}_i^{(1)}, \hat{A}\} + 2f_i \operatorname{Cov}\{\hat{a}_i^{(1)}, \hat{T}\} - 2f_i^2 \operatorname{Cov}\{\hat{A}, \hat{T}\}, \quad (36)$$

where

$$f_i = \frac{a_i}{\sum_{j=1}^{N} a_j}. \quad (37)$$

Since the trunk-group measurement $\hat{T}$ corresponds to a sampling rate of 100 percent (i.e., $p = 1$), we have

$$\hat{T} = \hat{A}|_{p=1} = \sum_{i=1}^{N} \hat{a}_i^{(1)}|_{p=1}. \quad (38)$$

Hence, from (21), (35), and (38), it follows that

$$E\{\hat{a}_i^{(3)}\} \approx a_i. \quad (39)$$

We assume that the daily source loads (for different point-pairs) are uncorrelated;* hence, the estimates $\hat{a}_i^{(1)}$ are uncorrelated. Furthermore, since our results are not significantly affected by differences in the mean holding times, we assume that each point-to-point load has the same mean holding time, h. Thus, we have

$$\operatorname{Var}\{\hat{A}\} = \sum_{i=1}^{N} \operatorname{Var}\{\hat{a}_i^{(1)}\} = \frac{1}{K} \sum_{i=1}^{N} \left\{ \frac{2a_i}{pt/h} + v_{di} \right\}, \quad (41)$$

$$\operatorname{Var}\{\hat{T}\} = \operatorname{Var}\{\hat{A}|_{p=1}\} = \frac{1}{K} \sum_{i=1}^{N} \left\{ \frac{2a_i}{t/h} + v_{di} \right\}, \quad (42)$$

and

$$\operatorname{Cov}\{\hat{a}_i^{(1)}, \hat{A}\} = \operatorname{Var}\{\hat{a}_i^{(1)}\} = \frac{1}{K} \left\{ \frac{2a_i}{pt/h} + v_{di} \right\}. \quad (43)$$

Also, from (32) and (38),

$$\operatorname{Cov}\{\hat{a}_i^{(1)}, \hat{T}\} = \operatorname{Cov}\{\hat{a}_i^{(1)}, \hat{a}_i^{(1)}|_{p=1}\} = \frac{1}{K} \left\{ \frac{2a_i}{t/h} + v_{di} \right\} \quad (44)$$

* We have shown that our results are independent of the covariance structure of the daily source loads; for simplicity, we assume that they are uncorrelated.

and

$$\operatorname{Cov}\{\hat{A},\hat{T}\} = \sum_{i=1}^{N} \operatorname{Cov}\{\hat{a}_i^{(1)}, \hat{T}\}$$

$$= \frac{1}{K}\sum_{i=1}^{N}\left\{\frac{2a_i}{t/h} + v_{di}\right\}. \tag{45}$$

Combining (26), (36), and (41) througb (45) gives

$$\operatorname{Var}\{\hat{a}_i^{(3)}\} \approx \frac{1}{K}\left\{\frac{2a_i}{pt/h} + v_{di}\right\} + \frac{2(1-p)}{Kpt/h} f_i^2 \sum_{j=1}^{N} a_j - \frac{4(1-p)}{Kpt/h} f_i a_i$$

or, since $f_i \sum_{j=1}^{N} a_j = a_i$,

$$\operatorname{Var}\{\hat{a}_i^{(3)}\} \approx \frac{2a_i}{Kpt/h}\{1 - f_i(1-p)\} + \frac{1}{K} v_{di}. \tag{46}$$

APPENDIX B

Minimum Variance Estimate

In this appendix, we show that Estimate 3 can be obtained as an approximation to a minimum variance linear estimate. Thus, from eq. (13)

$$\hat{a}_i^{(3)} = \hat{a}_i^{(1)} + w_i\{\hat{T} - \hat{A}\}. \tag{47}$$

This estimate is unbiased, i.e., $E\{\hat{a}_i^{(3)}\} = E\{\hat{a}_i^{(1)}\} = a_i$, and has variance

$$\operatorname{Var}\{\hat{a}_i^{(3)}\} = \operatorname{Var}\{\hat{a}_i^{(1)}\} + 2w_i \operatorname{Cov}\{\hat{a}_i^{(1)}, \hat{T} - \hat{A}\} + w_i^2 \operatorname{Var}\{\hat{T} - \hat{A}\}. \tag{48}$$

The value of w_i which minimizes the variance satisfies the equation

$$\frac{\partial \operatorname{Var}\{\hat{a}_i^{(3)}\}}{\partial w_i} = 0, \tag{49}$$

which implies that

$$w_i = \frac{\operatorname{Cov}\{\hat{a}_i^{(1)}, \hat{A} - \hat{T}\}}{\operatorname{Var}\{\hat{T} - \hat{A}\}}. \tag{50}$$

From Appendix A, it follows that

$$w_i = \frac{a_i}{\sum_{j=1}^{N} a_j}. \tag{51}$$

Now if a_i is estimated by $\hat{a}_i^{(1)}$ so that w_i is estimated by $\hat{a}_i^{(1)}/\hat{A}$, eq. (47) becomes

$$\hat{a}_i^{(3)} = \frac{\hat{a}_i^{(1)}}{\hat{A}} \hat{T}. \tag{52}$$

Q.E.D.

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